Trigonometric functions

e-Twinning Project

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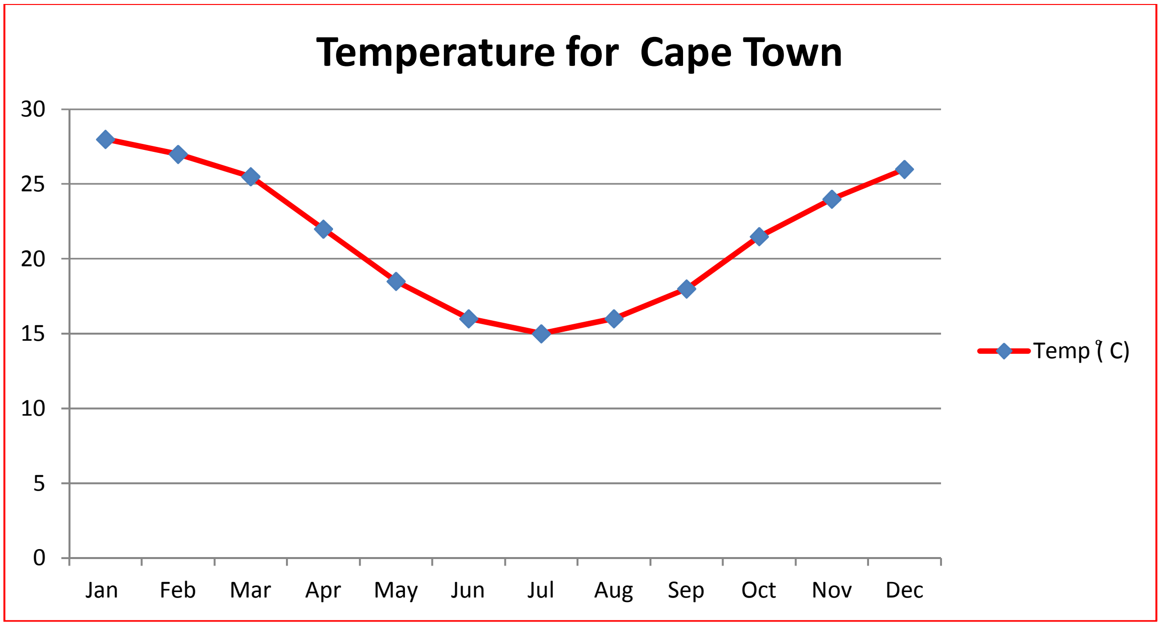
1. Mean Monthly Temperature

The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

We plot the temperature T on the vertical axis.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will repeat itself for the next 12 month period.

a) Find a sine model for the height of the tide H in terms of the time t hours after mean tide.

b) Sketch the graph of the model over one period, two period.

R.

The general cosine function is **y = a cos bx + c**, where a is amplitude, p=$\frac{360°}{|b|}$ is period, c translate the graph up or down.

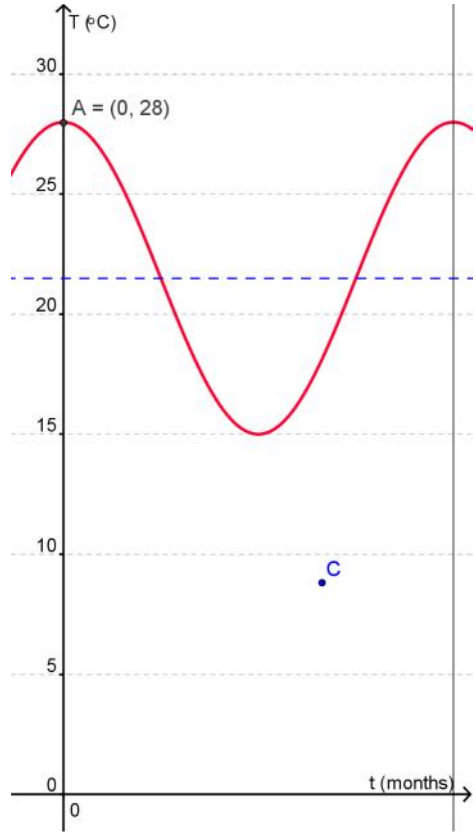
Since the maximum is at the start of the year, we attempt to model this data using the general cosine function **y = a cos bx + c**, or in this case **T(t) = a cos bt + c.**

The period is 12 months, so p=$\frac{360°}{|b|} = 12 \implies$ **b=30°.**

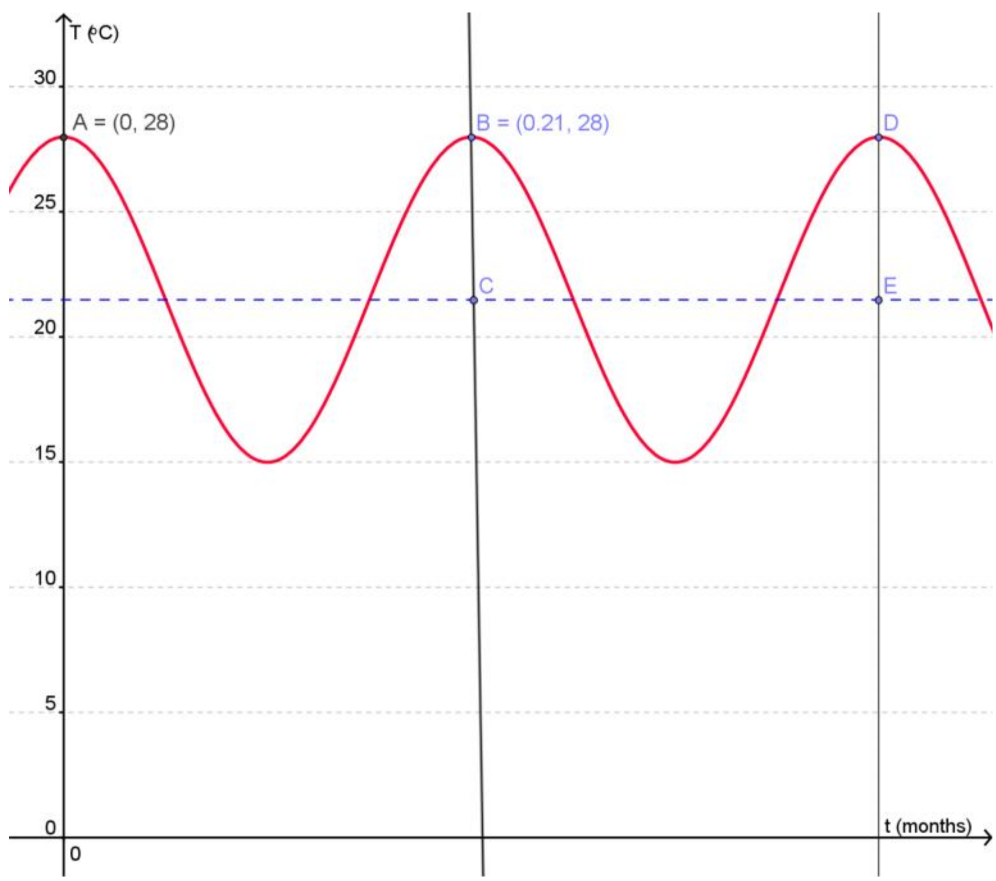
The amplitude is $a = \frac{T_{max} - T_{min}}{2} \implies a$**=6.5**

$c = \frac{T_{max} + T_{min}}{2} \approx 21.5 \implies c \approx 21.5$

The model is **T =6.5 cos(30t)+21.5** and is superimposed on the graph which follows:



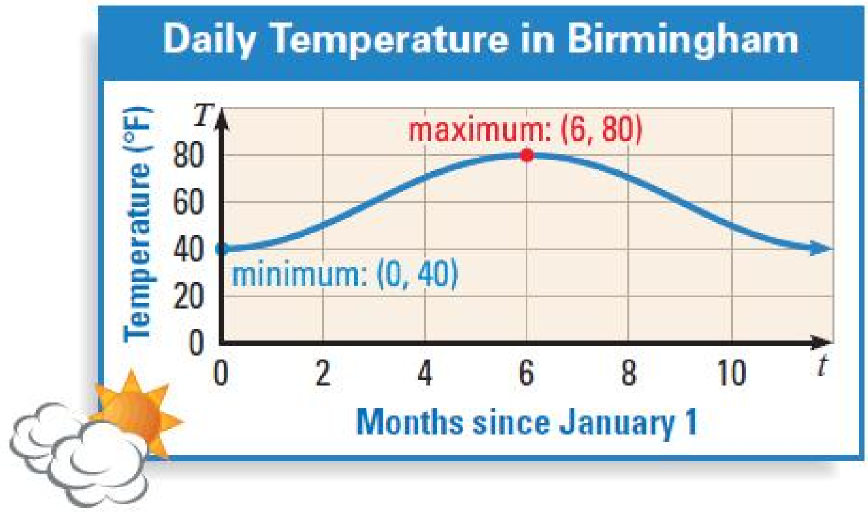
The graph over a two years period is shown below:



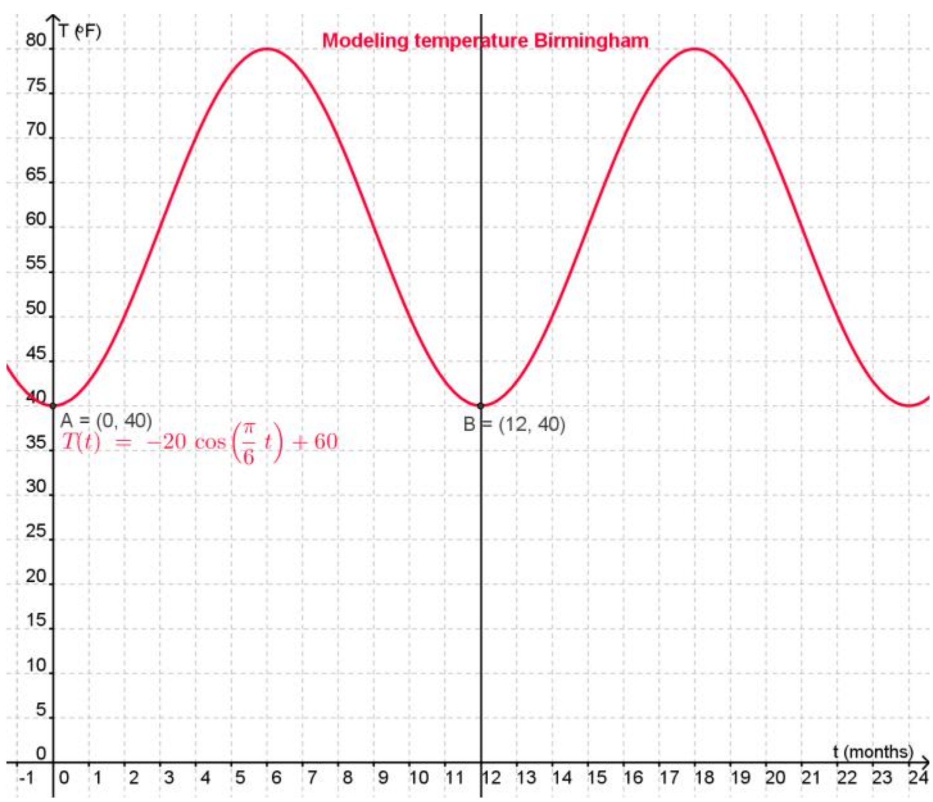
2. **Daily temperature in Birmingham**

Write a trigonometric model for the average daily temperature in Birmingham, Alabama.

Source: National Climatic Data Center



The graph crosses the *T*-axis at the minimum point. So if we model the temperature curve with a cosine function, there is a reflection but no horizontal shift.
The mean of the maximum and minimum values is 60, so there is a vertical shift of $k = 60$.
The period is $p = \frac{360°}{|b|} = 12 \implies$ **b=30º.**
The amplitude is $|a| = 20$, and because the graph is a reflection it follows that ***a* = –20**.
The model is T = **T =–20 cos(30t)+60** where t is measured in months and $t = 0$ represents January 1.



Simple Harmonic Motion

An object moving along the x-axis is said to exhibit **simple harmonic motion** if its position as a function of time varies as

$$\mathbf{y = a\cos(\omega(t-c)) + b.}$$

The object oscillates about the equilibrium position $x_0=b$. If we choose the origin of our coordinate system such that $x_0 = 0$, then the displacement x from the equilibrium position as a function of time is given by **$x(t) = a\cos(\omega(t-c))$**

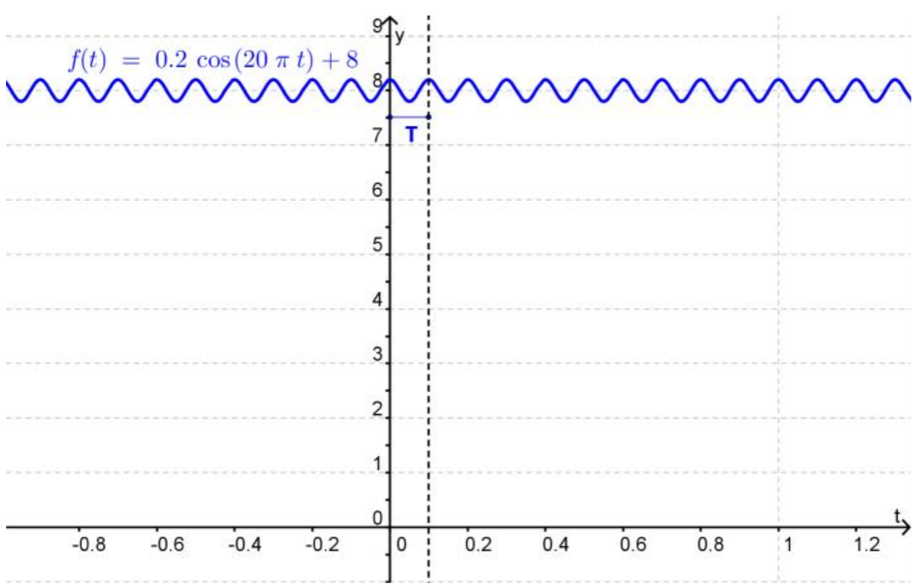
A is the **amplitude** of the oscillation, i.e. the maximum displacement of the object from equilibrium, either in the positive or negative x-direction. Simple harmonic motion is repetitive. The **period** T is the time it takes the object to complete one oscillation and return to the starting position. The **angular frequency** ω is given by $\omega = 2\pi/T$. The angular frequency is measured in radians per second. The inverse of the period is the **frequency** $f = 1/T$. The frequency $f = 1/T = \omega/2\pi$ of the motion gives the number of complete oscillations per unit time. It is measured in units of Hertz, (1Hz = 1/s).

1. **A Bobbing Cork**
   A cork floating in a lake is bobbing in simple harmonic motion. Its displacement above the bottom of the lake is modeled by **$y=0.2\cos 20\pi t+8$**, where **y** is measured in meters and **t** is measured in minutes.
   a) Find the frequency of the motion of the cork.
   b) Sketch a graph of y.
   c) Find the maximum displacement of the cork above the lake bottom.
   R.

a) The function has the form: **$y=0.2\cos 20\pi t+8$** $\Rightarrow \omega=20\pi$

The period is: $T = \frac{2\pi}{\omega} = \frac{2\pi}{20\pi} = \frac{1}{10} \Rightarrow$ the frequency is: $f = \frac{1}{T} = 10$.

b) Graph is below:



c) The amplitude is a=0.2.

2. **Modeling the Brightness of a Variable Star**

A variable star is one whose brightness alternately increases and decreases. For the variable star Delta Cephei, the time between periods of maximum brightness is 5.4 days. The average brightness (or magnitude) of the star is 4.0, and its brightness varies by ±0.35 magnitude.

a) Find a function that model the brightness of Delta Cephei as a function of time.
b) Sketch a graph of the brightness of Delta Cephei as a function of time.

R.

a) The form of the function is **y=acos(ω(t–c)) +b**

The amplitude is the maximum variation from average brightness, so the amplitude is a=0.35 magnitude. The period is 5.4 days, so $\omega = \frac{2\pi}{5.4} \approx 1.164$.

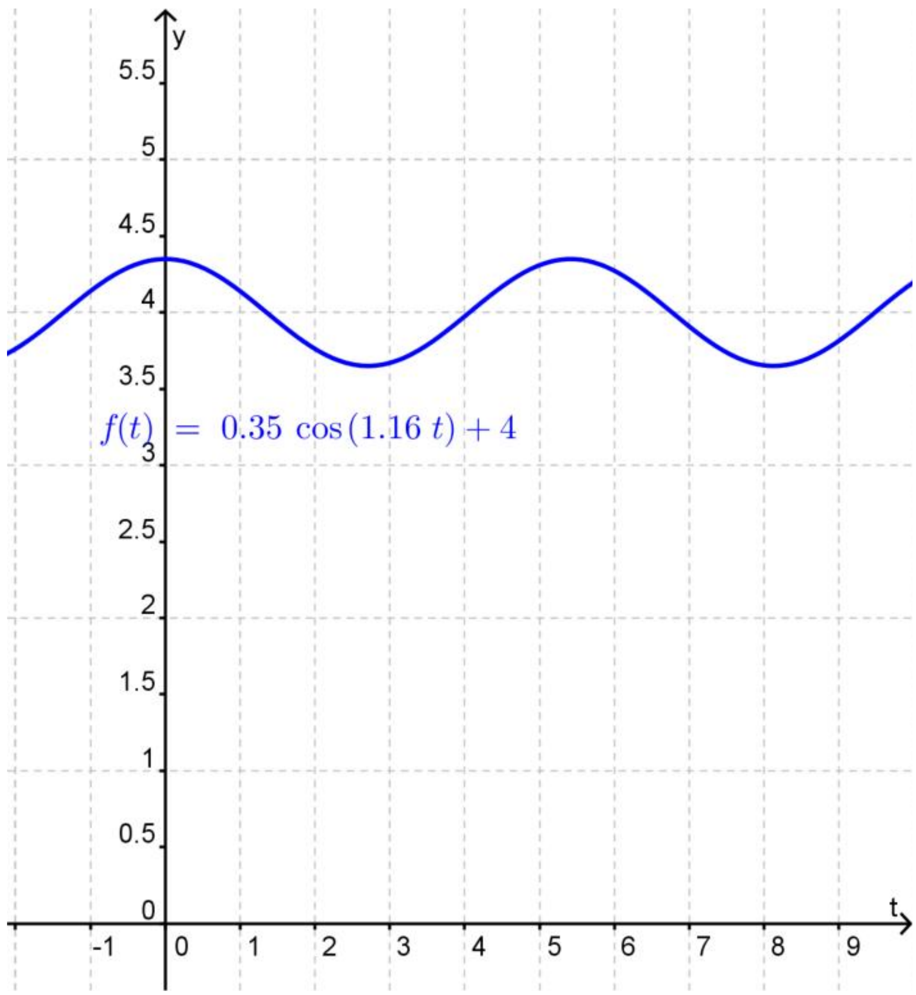
Since the brightness varies from an average value of 4.0 magnitudes, the graph is shifted upward by b=4.

If we take t=0 to be a time when the star is at maximum brightness, there is no horizontal shift, so c=0 (because a cosine curve achieves its maximum at t=0). Thus the function is

y=0.35cos(1.16t) +4

where t is the number of days from a time when the star is at maximum brightness.

c) The graph is :



3. **Blood Pressure**

Each time your heart beats, your blood pressure increases, then decreases as the heart rests between beats. A certain person's blood pressure is modeled by the function **p(t)= 115+25sin(160πt)**, where **p(t)** is the pressure in mmHg at time **t**, measured in minutes.

a) Find the amplitude, period, and frequency of p.

b) Sketch a graph of p.

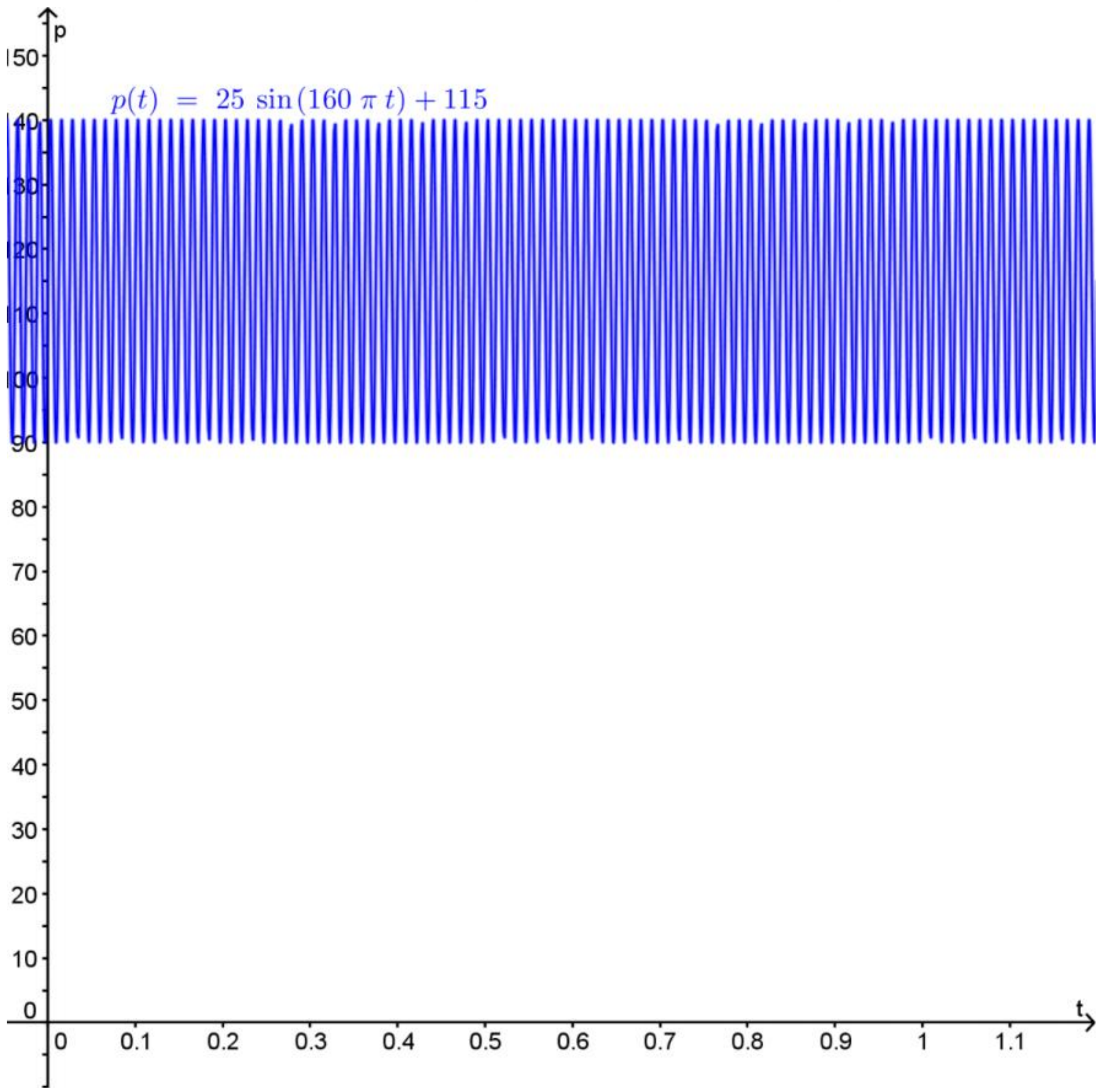
c) If a person is exercising, his or her heart beats faster. How does this affect the period and frequency of p?

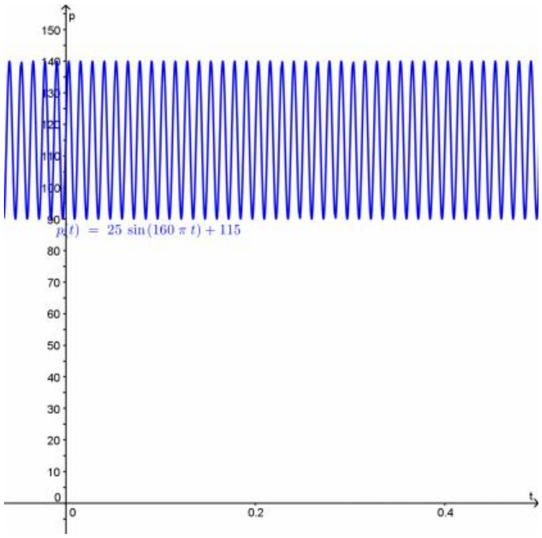
R.

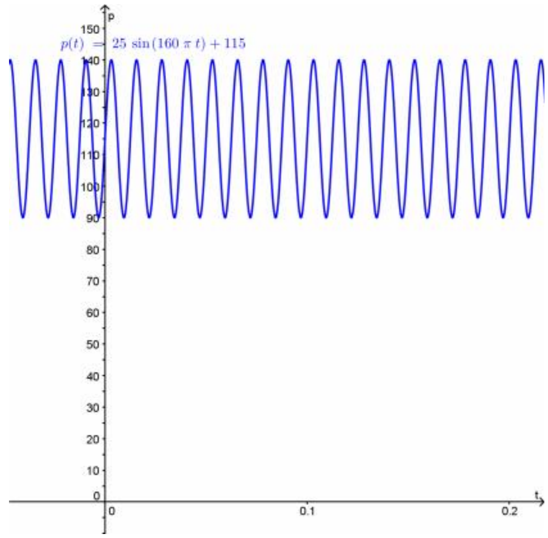
The function has the form: **p(t)= 115+25sin(160πt)** $\Longrightarrow \omega=160\pi$

a) The amplitude is a=25.

b) The period is: $T = \frac{2\pi}{\omega} = \frac{2\pi}{160\pi} = \frac{1}{80} \Longrightarrow$ the frequency is: $f = \frac{1}{T} = 80$.







Damped harmonic motion

If the equation describing the displacement y of an object at time t is
$y=ke^{-ct}\sin\omega t$ or $y=ke^{-ct}\cos\omega t$, $c > 0$
then the object is in damped harmonic motion. The constant ***c*** is the damping constant, ***k*** is the initial amplitude, and $2\pi/w$ is the period.

Two mass-spring systems are experiencing damped harmonic motion, both at 0.5 cycles per second and both with an initial maximum displacement of 10 cm. The first has a damping constant of 0.5 and the second has a damping constant of 0.1.

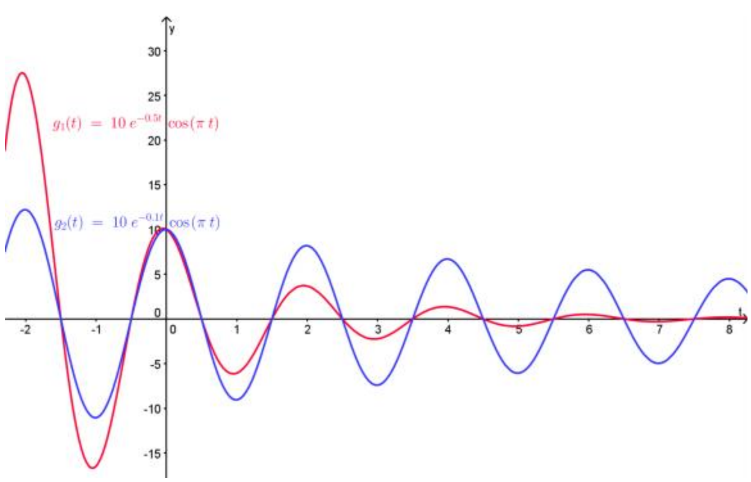
a) Find functions of the form $g(t)=ke^{-ct}\cos\omega t$ to model in each case.
b) Graph the two functions you found in part a). How do they differ?

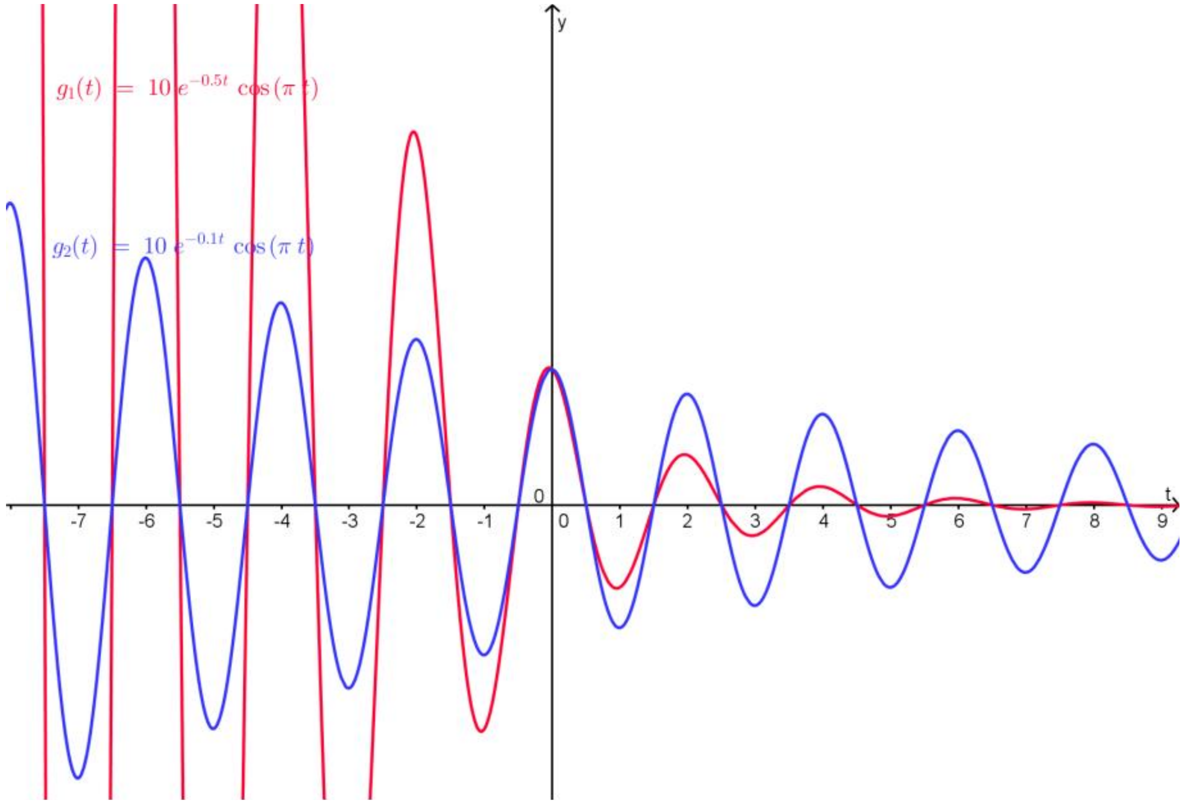
R. a) At time t=0 the displacement is 10. Thus, $g(0)= ke^{-ct}\cos(\omega\cdot 0)=k$, so k=10

Also, the frequency is f=0.5Hz, and since $\omega=2\pi f$, we get $\omega=2\pi(0.5)= \pi$

Using the given damping constants, we find that the motions of two spring are given by the functions:

$$g_1(t)= 10e^{-0.5t}\cos\pi t \quad \text{and} \quad g_2(t)= 10e^{-01t}\cos\pi t$$





4. **Guitar string**

   A guitar string is pulled at point P a distance of 3 cm above its rest position. It is then released and vibrates in damped harmonic motions with a frequency of 165 cycles per second. After 2 s, it is observed that the amplitude of the vibration at point P is 0.5 cm.

   a) Find the damping constant c.

   b) Find an equation that descibes the position of point P above its rest position as a function of time. Take t=0 to be the instant that the string is released.

   R.

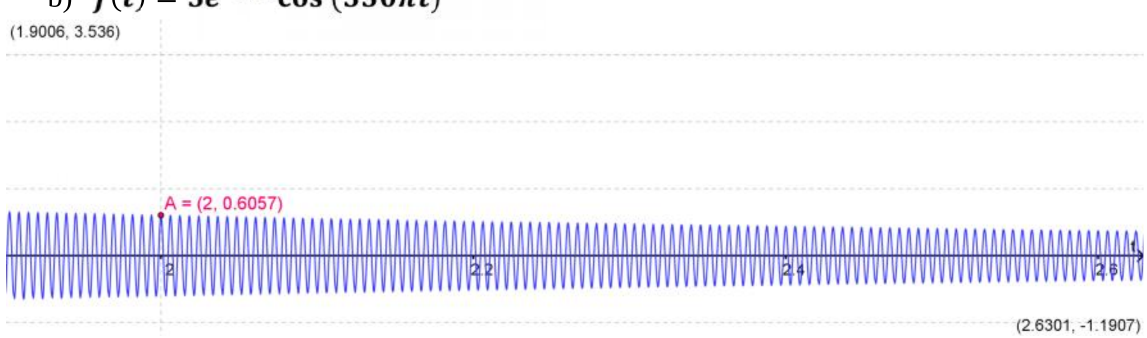
   $y = ke^{-ct}\cos\omega t$, $c > 0$

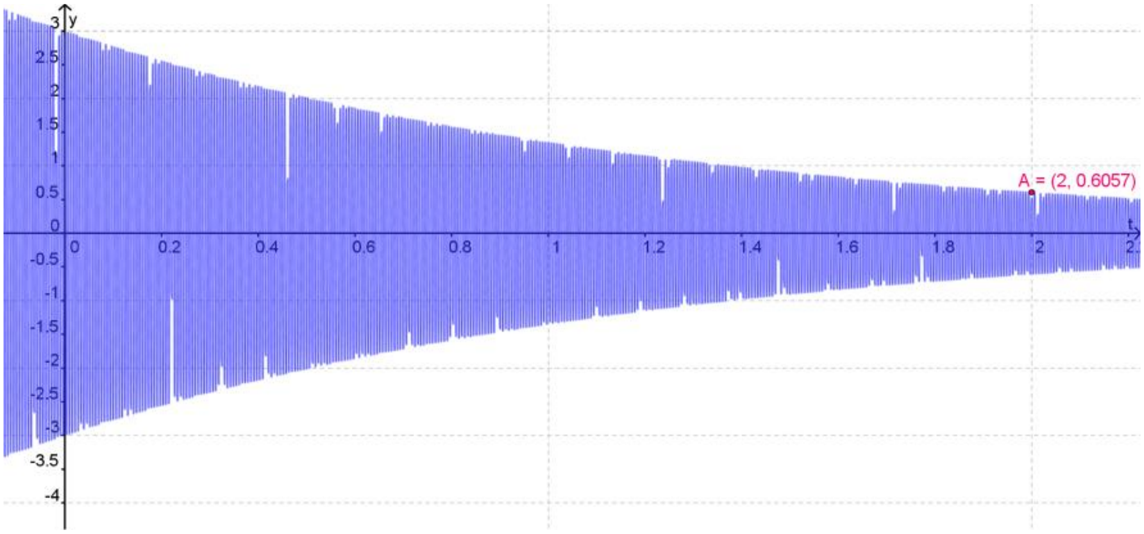
a) In point P, for t=0 the displacement is 3. Thus, $y(0) = ke^{-ct}\cos(\omega\cdot 0) = k$, so k=3.

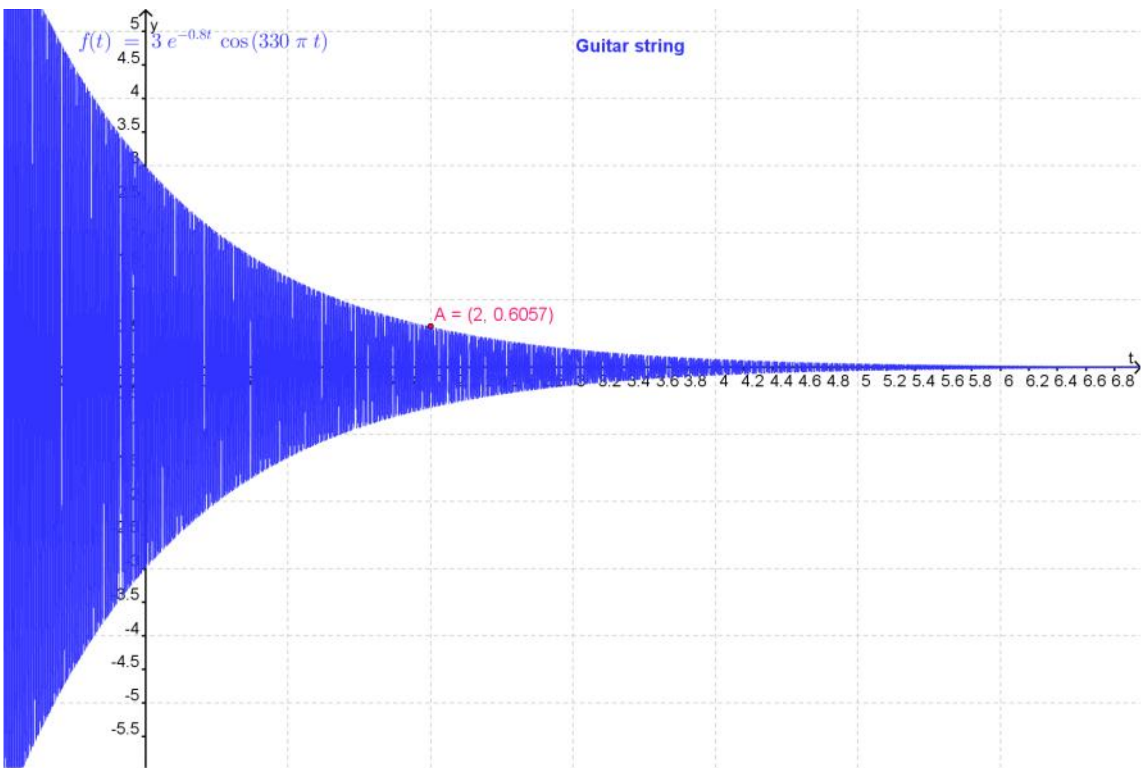
$$3e^{-2c} = 0.6 \Longrightarrow e^{-2c} = 0.2 \Longrightarrow -2c = \ln(0.2) \Longrightarrow 2c = \ln 5 \Longrightarrow c = \frac{\ln 5}{2} \Longrightarrow \mathbf{c = 0.8}$$

is f=165Hz, and since $\omega = 2\pi f$, we get $\boldsymbol{\omega} = 2\pi\cdot 165 = \mathbf{330\pi}$

b) $\boldsymbol{f(t) = 3e^{-0.8t}\cos(330\pi t)}$







Shock Absorber

When a car hits a certain bump on the road, a shock absorber on the car is compressed a distance of 6 in., then released (see the figure). The shock absorber vibrates in damped harmonic motion with a frequency of 2 cycles per second. The damping constant for this particular shock absorber is 2.8.

a) Find an equation that describes the

displacement of the shock absorber from its rest position as a function of time. Take t=0 to be the instant that the shock absorber is released.

b) Graph the function you found in part a).

c) How long does it take for the amplitude of the vibratio: to decrease to 0.4 in?

R.

a) $y = ke^{-ct}\cos\omega t$, $c > 0$

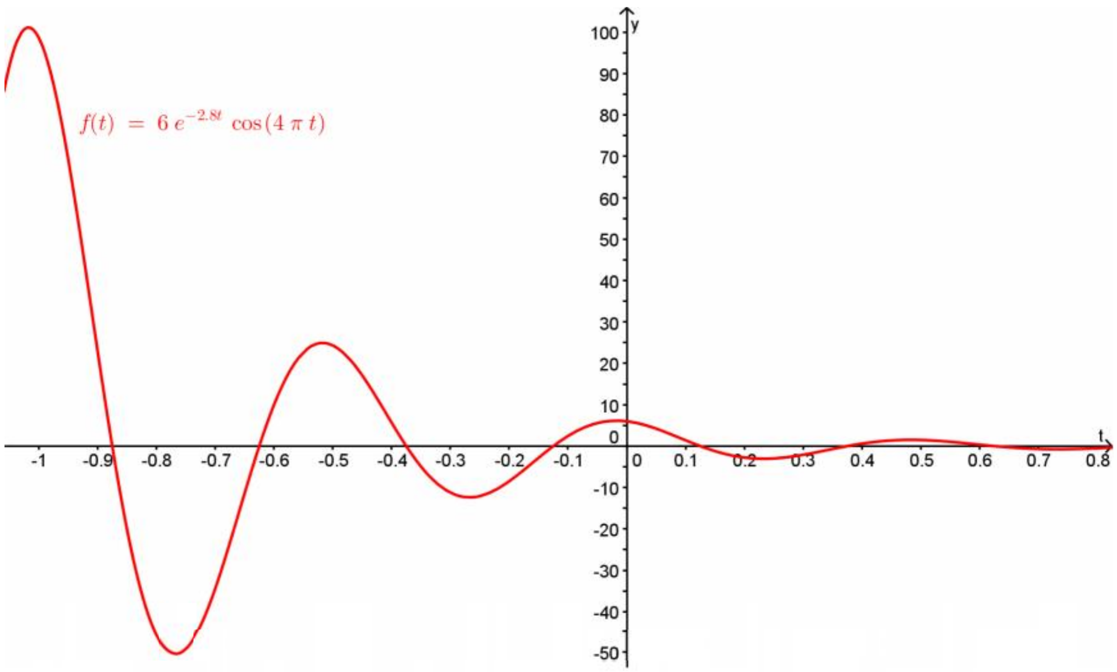
$c = 2.8$. $y = ke^{-ct}\cos\omega t$, $c > 0$ $y = ke^{-ct}\cos\omega t$

For t=0 the displacement is 6. Thus, $y(0) = ke^{-ct}\cos(\omega \cdot 0) = k$, so k=6.

f=2 Hz, and since $\omega = 2\pi f$, we get $\omega = 2\pi \cdot 2 = 4\pi$. $\omega = 4\pi$

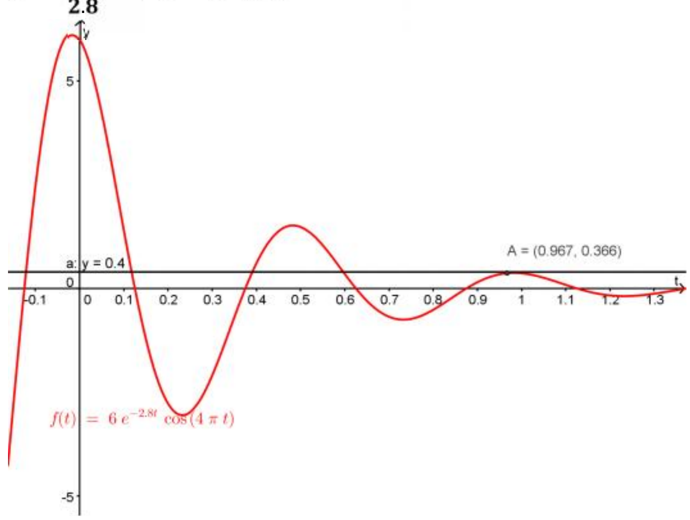
$$y = 6e^{-2.8t}\cos(4\pi t)$$

b) The graph is:



c) $f(t) = 6e^{-2.8t}\cos(4\pi t) \Rightarrow 0.4 = 6e^{-2.8t} \Rightarrow e^{-2.8t} = \frac{1}{15} \Rightarrow e^{2.8t} = 15 \Rightarrow$

$2.8t = ln15 \Rightarrow t = \frac{ln15}{2.8} \Rightarrow t \approx 0.967$



Websites:

- http://books.google.ro/books?id=PacIAAAAQBAJ&printsec=frontcover&hl=ro#v=onepage&q&f=false
- http://web.utk.edu/~cnattras/Physics221Spring2013/modules/m3/harmonic_motion.htm
- http://www.classzone.com/eservices/home/pdf/student/LA214EAD.pdf